A. The intuitive approach (2.2):

1. Watch the video at:


2. Explain in your own words what is meant by the equation

\[ \lim_{x \to 2} f(x) = 5 \]

Is it possible for this statement to be true and yet \( f(2) = 3 \)? Explain.

3. Explain what it means to say that

\[ \lim_{x \to 1^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 7 \]

In this situation is it possible that \( \lim_{x \to 1} f(x) \) exists? Explain.

4. Below is the graph of a function \( f(x) \). For each of the given points determine the value of \( f(a) \) and of \( \lim_{x \to a} f(x) \). If any of these quantities do not exist, clearly explain why.

(a) \( a = -2 \) 
(b) \( a = -1 \) 
(c) \( a = 2 \) 
(d) \( a = 3 \)
5. Below is the graph of a function $f(x)$. For each of the given points determine the value of $f(a)$ and of $\lim_{x \to a} f(x)$. If any of these quantities do not exist, clearly explain why.

(a) $a = -3$  
(b) $a = -1$  
(c) $a = 1$  
(d) $a = 3$

6. Below is the graph of a function $f(x)$. For each of the given points determine the value of $f(a)$, of $\lim_{x \to a} f(x)$, $\lim_{x \to a^-} f(x)$ and of $\lim_{x \to a^+} f(x)$. If any of these quantities do not exist, clearly explain why.

(a) $a = -5$  
(b) $a = -2$  
(c) $a = 1$  
(d) $a = 4$
7.

Sketch a graph of a function that satisfies each of the following conditions.

\[ \lim_{x \to -5^-} f(x) = -1 \quad \lim_{x \to -5^+} f(x) = 7 \quad f(-5) = 4 \]
\[ \lim_{x \to 4} f(x) = 6 \quad f(4) \text{ does not exist} \]

8. Guess the value of the corresponding limit by using the table method. When possible, verify your result algebraically (that is, simplify \( f(x) \) and calculate the limit of the simpler expression) or graphically (if you have a graphing device):

a) \( \lim_{x \to 2} x \)

b) \( \lim_{x \to 2} x^2 \)

c) \( \lim_{x \to 2} x^2 \)

d) \( \lim_{x \to 2} \frac{x + 3}{x - 1} \)

e) \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \)

f) \( \lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x} \)

g) \( \lim_{x \to 0} \frac{e^{5x} - 1}{x} \)

h) \( \lim_{x \to 0} \frac{\sin(x)}{x} \)

i) \( \lim_{x \to 2} \frac{1}{(x - 2)^2} \)

j) \( \lim_{x \to 2} \frac{1}{x - 2} \)

k) \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \)
B. Limit laws (2.3):

1.

Given that

\[
\lim_{x \to 2} f(x) = 4 \quad \lim_{x \to 2} g(x) = -2 \quad \lim_{x \to 2} h(x) = 0
\]

find the limits that exist. If the limit does not exist, explain why.

(a) \( \lim_{x \to 2} [f(x) + 5g(x)] \)

(b) \( \lim_{x \to 2} [g(x)]^3 \)

(c) \( \lim_{x \to 2} \sqrt{f(x)} \)

(d) \( \lim_{x \to 2} \frac{3f(x)}{g(x)} \)

(e) \( \lim_{x \to 2} \frac{g(x)}{h(x)} \)

(f) \( \lim_{x \to 2} \frac{g(x)h(x)}{f(x)} \)

(g) \( \lim_{x \to 2} [(3 - f(x))(1 + 2g(x))] \)

(h) \( \lim_{x \to 2} \sqrt{3 + 6f(x) - h(x)} \)

2. For each of the following exercises, use the limit properties to calculate the limit. If it is not possible to calculate the limit, clearly explain why not:

a) \( \lim_{x \to a} (3x^2 - 9x + 2) \)

b) \( \lim_{x \to -2} \frac{x^4 - 2}{2x^2 - 3x + 2} \)

c) \( \lim_{x \to -2} \sqrt{x^4 + 3x + 6} \)

3. Evaluate the limit, if it exists:

a) \( \lim_{x \to 0} \frac{x^4 + 6}{x^3 - 3} \)

b) \( \lim_{x \to -2} \frac{x + 2}{x^2 - 6x - 16} \)
c) \[ \lim_{t \to -3} \frac{t^2 - 9}{2t^2 + 7t + 3} \]

d) \[ \lim_{h \to 0} \frac{(2 + h)^3 - 8}{h} \]

e) \[ \lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1} \]

f) \[ \lim_{x \to 2} \frac{\sqrt{4x+1}-3}{x-2} \]

g) \[ \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \]

h) \[ \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2 + x}\right) \]

i) \[ \lim_{x \to 16} \frac{4 - \sqrt{x}}{16x - x^2} \]

4.

Let

\[ g(x) = \begin{cases} 
  x & \text{if } x < 1 \\
  3 & \text{if } x = 1 \\
  2 - x^2 & \text{if } 1 < x \leq 2 \\
  x - 3 & \text{if } x > 2 
\end{cases} \]

(a) Evaluate each of the following, if it exists.
(i) \[ \lim_{x \to 1^-} g(x) \]
(ii) \[ \lim_{x \to 1^+} g(x) \]
(iii) \[ g(1) \]
(iv) \[ \lim_{x \to 2^-} g(x) \]
(v) \[ \lim_{x \to 2^+} g(x) \]
(vi) \[ \lim_{x \to 2} g(x) \]

(b) Sketch the graph of \( g \).

5. Find the limit, if it exists. If the limit does not exist, explain why:

a) \[ \lim_{x \to 3} (2x + |x-3|) \]

b) \[ \lim_{x \to 6} \frac{2x+12}{|x+6|} \]

c) \[ \lim_{x \to 2} \frac{2-|x|}{2+x} \]
6.

Let
\[ f(x) = \begin{cases} 
  x^2 + 1 & \text{if } x < 1 \\
  (x - 2)^2 & \text{if } x \geq 1 
\end{cases} \]

(a) Find \( \lim_{x \to 1^-} f(x) \) and \( \lim_{x \to 1^+} f(x) \).
(b) Does \( \lim_{x \to 1} f(x) \) exist?
(c) Sketch the graph of \( f \).

7. Given the function:
\[ f(x) = \begin{cases} 
  x^2 - x^3 & \text{for } x < 2 \\
  5x - 14 & \text{for } x \geq 2 
\end{cases} \]
evaluate the following limits, if they exist:

a) \( \lim_{x \to 3^-} f(x) \)

b) \( \lim_{x \to 2} f(x) \)

8.

Evaluate \( \lim_{x \to 2} \frac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1} \).

9.

Is there a number \( a \) such that
\[ \lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2} \]
eexists? If so, find the value of \( a \) and the value of the limit.
C. The Rigorous Definition of Limit (2.4) (Optional):

1. Watch the videos at:


and


2. Explain as precisely (as rigorously) as you can what each of the following means and illustrate with a sketch:

\[
\begin{align*}
(a) \lim_{x \to a} f(x) &= L \\
(b) \lim_{x \to a^+} f(x) &= L \\
(c) \lim_{x \to a^-} f(x) &= L \\
(d) \lim_{x \to a} f(x) &= \infty \\
(e) \lim_{x \to \infty} f(x) &= L
\end{align*}
\]

3. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

4. Determine whether the following statement is true or false. If it is true, explain why. If it is false, explain why or give an example which disproves the statement:

Let \( f(x) \) be a function such that \( \lim_{x \to 0} f(x) = 6 \). Then it is true that there exists a number \( \delta > 0 \) such that if

\[0 < |x| < \delta \text{ then } |f(x) - 6| < 1.\]
5. Use the given graph of \( f \) to find a number \( \delta \) such that

\[
\text{if } |x - 1| < \delta \quad \text{then} \quad |f(x) - 1| < 0.2
\]

What does this graph and your calculations suggest about \( \lim_{{x \to 1}} f(x) \) ?

6. Use the given graph of \( f \) to find a number \( \delta \) such that

\[
\text{if } 0 < |x - 3| < \delta \quad \text{then} \quad |f(x) - 2| < 0.5
\]

What does this graph and your calculations suggest about \( \lim_{{x \to 3}} f(x) \) ?
7.

Use the given graph of \( f(x) = x^2 \) to find a number \( \delta \) such that

\[
| x - 1 | < \delta \quad \text{then} \quad | x^2 - 1 | < \frac{1}{2}
\]

What does this graph and your calculations suggest about \( \lim_{x \to 1} x^2 \)?

8. Prove the following statements using the rigorous \( \varepsilon - \delta \) definition of a limit:

a) \( \lim_{x \to 1} \frac{2 + 4x}{3} = 2 \)

b) \( \lim_{x \to 1} \left( 3 - \frac{4}{5} x \right) = -5 \)

c) \( \lim_{x \to a} x = a \)

d) \( \lim_{x \to 0} x^2 = 0 \)

e) \( \lim_{x \to 2} \left( x^2 + 2x - 7 \right) = 1 \)
D. Continuity (2.5):

1. Write an equation that expresses the fact that a function \( f(x) \) is continuous at the number 4.

2. If \( f(x) \) is continuous on \((-\infty, \infty)\), what can you say about its graph?

3. Sketch the graph of a function \( f(x) \) that is continuous except at the stated discontinuity:
   a) discontinuous at -1 and 4, but continuous from the left at -1 and from the right at 4;
   b) has a removable discontinuity at 3 and a non-removable (jump) discontinuity at 5.

5. The graph of a function \( f(x) \) is shown below. Based on this graph, determine where the function is discontinuous:
   a)
6. Use the definition of continuity and the methods to calculate limits to determine if the given function is continuous or discontinuous at the given points. Support your work:

a) 

\[ f(x) = \frac{6 + 2x}{7x - 14} \]

(a) \( x = -3 \), (b) \( x = 0 \), (c) \( x = 2 \)
b) 

\[ h(z) = \begin{cases} 
2z^2 & z < -1 \\
4z + 6 & z \geq -1 
\end{cases} \]

(a) \( z = -6 \), (b) \( z = -1 \)?

c) 

\[ Z(t) = \begin{cases} 
8 & t < 5 \\
1 - 6t & t \geq 5 
\end{cases} \]

(a) \( t = 0 \), (b) \( t = 5 \)?

d) 

\[ h(z) = \begin{cases} 
z + 2 & z < -4 \\
0 & z = -4 \\
18 - z^2 & z > -4 
\end{cases} \]

(a) \( z = -4 \), (b) \( z = 2 \)?

e) 

\[ f(x) = \begin{cases} 
\cos x & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
1 - x^2 & \text{if } x > 0 
\end{cases} \]

7. Determine where the given function is discontinuous:

a) \( f(x) = \frac{11 - 2x}{2x^2 - 13x - 7} \) 

b) \( f(x) = \frac{x^2 - 1}{x^3 + 6x^2 + x} \)
c) \( f(x) = \frac{4x + 1}{\cos\left(\frac{x}{2}\right) + 1} \)

\[ d) \quad f(x) = \begin{cases} 1 + x^2 & \text{if } x \leq 0 \\ 2 - x & \text{if } 0 < x \leq 2 \\ (x - 2)^3 & \text{if } x > 2 \end{cases} \]

8. Find \( a \in R \) such that the function \( f(x) = \begin{cases} x + 3 & \text{if } x \leq 3 \\ a \cdot x & \text{if } x > 3 \end{cases} \) is continuous for \( x = 3 \).

9. For what value of the constant \( c \) is the function \( f \) continuous on \((-\infty, \infty)\)?

\[ f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases} \]

10. Find the values of \( a \) and \( b \) that make \( f \) continuous everywhere.

\[ f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases} \]

11. Use the Intermediate Value theorem to show that there is at least one root of the specified equation in the given interval. Support your work by showing the continuity of the appropriate function:

a) \( 1 + 7x^3 - x^4 = 0 \) on \([4,8]\)

b) \( x^2 + 11x = 3 \) on \([-15, -5]\)

c) \( \frac{x^2 + x - 15}{x - 8} = 0 \) on \([-5,1]\)

d) \( \ln(2x^2 + 1) - \ln(x^2 - 4) = 0 \) on \([-1,2]\)

e) \( 10 = x^3 + x^2 e^{-x} - 5 \) on \([0,4]\)
f) \( e^x = 3 - 2x \) on \([0,1]\)

g) \( \sin(x) = x^2 - x \) on \([1,2]\)

E. Limits at infinity and infinite limits. Horizontal Asymptotes and Vertical asymptotes. (2.6 and 2.2).

1. Explain as precisely as you can the meaning of:

a) \( \lim_{x \to -3} f(x) = \infty \)

b) \( \lim_{x \to -4^+} f(x) = -\infty \)

c) \( \lim_{x \to \infty} f(x) = 5 \)

d) \( \lim_{x \to -\infty} f(x) = 3 \)

2. For the function \( R \) whose graph is shown, state the following.

(a) \( \lim_{x \to -2} R(x) \)

(b) \( \lim_{x \to 5} R(x) \)

(c) \( \lim_{x \to -3} R(x) \)

(d) \( \lim_{x \to -3^+} R(x) \)

(e) The equations of the vertical asymptotes.
3. For the function $f$ whose graph is shown, state the following.
   (a) $\lim_{x \to -7} f(x)$
   (b) $\lim_{x \to -3} f(x)$
   (c) $\lim_{x \to 0} f(x)$
   (d) $\lim_{x \to 6^+} f(x)$
   (e) $\lim_{x \to 6^-} f(x)$
   (f) The equations of the vertical asymptotes.

4. For the function $f$ whose graph is given, state the following.
   (a) $\lim_{x \to \infty} f(x)$
   (b) $\lim_{x \to -\infty} f(x)$
   (c) $\lim_{x \to 1} f(x)$
   (d) $\lim_{x \to 3} f(x)$
   (e) The equations of the asymptotes

5. Calculate the following infinite limits. Support your work:
   a) $\lim_{x \to -3} \frac{x+2}{x+3}$
   b) $\lim_{x \to -3} \frac{x+2}{x+3}$
   c) $\lim_{x \to 3} \frac{x+2}{x+3}$
   d) $\lim_{x \to -2} \frac{x^2 - 2x - 8}{x^2 - 5x + 6}$
   e) $\lim_{x \to -2} \frac{x^2 - 2x - 8}{x^2 - 5x + 6}$
   f) $\lim_{x \to -2} \frac{1+x}{x^3 + 8}$
6. Find the equations of all the vertical asymptotes for the given function:

a) \( f(x) = \frac{-6}{9-x} \)  
   b) \( f(x) = \frac{x^2+1}{3x-2x^2} \)

c) \( f(x) = \frac{\sin(4x)}{x} \)  
   d) \( f(x) = \frac{\tan(4x)}{x} \)

7. Calculate the following limits. Support your work:

a) \( \lim_{x \to \infty} \frac{3x-2}{2x+1} \)  
   b) \( \lim_{x \to \infty} \frac{x-2}{x^2+1} \)

c) \( \lim_{x \to \infty} \frac{(2x^2+1)^2}{(x-1)^2(x^2+x)} \)  
   d) \( \lim_{x \to -\infty} \frac{1}{x} \)

e) \( \lim_{x \to 1} \frac{x^2-1}{x-1} \)

f) \( \lim_{x \to 2} \frac{\sqrt{x+2}-2}{x-2} \)

g) \( \lim_{x \to \frac{\pi}{4}} \frac{\cos(x) - \sin(x)}{\cos(2x)} \)

h) \( \lim_{x \to \infty} \frac{2x^4+3x-1}{x^3-x+1} \)

i) \( \lim_{x \to \infty} (x+\sqrt{x^2+2x}) \)

j) \( \lim_{x \to \infty} (\sqrt{x^2+1}) \)

k) \( \lim_{x \to \infty} (\sqrt{x^2+1}) \)

l) \( \lim_{x \to \infty} \frac{(x+1)^3-x^3}{x^2} \)

m) \( \lim_{x \to \infty} (x^2-x) \)

n) \( \lim_{x \to \infty} \frac{\sin^2(x)}{x^2+1} \)

8. Find the horizontal asymptotes of each curve:

a) \( f(x) = \frac{2x+1}{x-2} \)
   b) \( f(x) = \frac{1+x^4}{x^2-x^4} \)

c) \( f(x) = \frac{\sqrt{2x^2+1}}{3x-5} \)
F. Derivatives and Rates of Change (2.7):

1. Watch the video at:

2. Solve 10 exercises from:

3. Write an expression for the slope of the tangent line to the curve \( y = f(x) \) at the point \((a, f(a))\).

4. Suppose that an object moves along a straight line with position \( f(t) \) at time \( t \). Write an expression for the instantaneous velocity of the object at time \( t = a \). How can you interpret this velocity in terms of the graph of \( f(t) \)?

5. Find the slope of the tangent line to the parabola \( y = 4x - x^2 \) at the point \((1,3)\). Find an equation of this tangent line.

6. Find the slope of the tangent line to the curve \( y = x - x^3 \) at the point \((1,0)\). Find an equation of this tangent line.

7. Find an equation of the tangent line to the curve \( y = \frac{2x+1}{x+2} \) at the point \((1,1)\).

8. a) Find the slope of the tangent line to the curve \( y = 3 + 4x^2 - 2x^3 \) at the point where \( x = a \).

b) Using part a), find equations of the tangent lines at the points \((1,5)\) and \((2,3)\).

c) Using a graphical device, graph the curve and both tangents on a common screen.
9.
(a) A particle starts by moving to the right along a horizontal line; the graph of its position function is shown. When is the particle moving to the right? Moving to the left? Standing still?
(b) Draw a graph of the velocity function.

![Graph of position function](image1)

10.
Shown are graphs of the position functions of two runners, A and B, who run a 100-m race and finish in a tie.

![Graph of position functions for runners A and B](image2)

(a) describe and compare how the runner run the race in terms of their instantaneous speeds.

b) at what time is the distance between the runners the greatest?

c) at what time do they have the same instantaneous speeds?
11. If a ball is thrown into the air with an initial velocity of 40 ft/sec, its height (in feet) after \( t \) seconds is given by:

\[
y(t) = 40t - 16t^2.\]

Find its velocity when \( t = 2 \).

12. Find \( f'(a) \) for:

a) \( f(x) = 3x^2 - 4x + 1 \)  

b) \( f(x) = 2x^3 + x \)

c) \( f(x) = x^{-2} \)

**G. The derivative as a function (2.8):**

1. Write a formula for the derivative \( f'(x) \) of a function \( f(x) \). Can you write another formula for \( f'(x) \)?

2. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.
3. The graph of a function $f(x)$ is given. For each function, sketch the graph of its derivative $f'(x)$ directly underneath it:

a)

![Graph](image1)

b)

![Graph](image2)

c)

![Graph](image3)

d)

![Graph](image4)

4. Calculate $f'(x)$ using your definition from exercise 1. State the domain of $f(x)$ and the domain of $f'(x)$ in each case:

a) $f(x) = \frac{1}{2}x - \frac{1}{3}$

b) $f(x) = \sqrt{x}$
c) \( f(x) = \frac{1}{\sqrt{x}} \)  

\[ d) \quad y = \frac{1 - 2x}{3 + x} \]

5.

a) What does it mean for \( f(x) \) to be differentiable at \( a \)?

b) What is the relationship between differentiability and continuity for a function?

c) Sketch the graph of a function which is continuous but not differentiable at \( a = 2 \).

d) Describe several ways in which a function can fail to be differentiable. Illustrate each case with sketches.

6. The graph of \( f(x) \) is given. In each case, state with reasons all points at which \( f(x) \) is not differentiable:

a)

b)

c)

d)
The left-hand and right-hand derivatives of \( f \) at \( a \) are defined by

\[
f'_-(a) = \lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}
\]

and

\[
f'_+(a) = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}
\]

if these limits exist. Then \( f'(a) \) exists if and only if these one-sided derivatives exist and are equal.

(a) Find \( f'_-(4) \) and \( f'_+(4) \) for the function

\[
f(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
5 - x & \text{if } 0 < x < 4 \\
\frac{1}{5 - x} & \text{if } x \geq 4 
\end{cases}
\]

(b) Sketch the graph of \( f \).
(c) Where is \( f \) discontinuous?
(d) Where is \( f \) not differentiable?